Long Time Propagation of Chaos for mean-field SDEs

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Outline

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Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space with the filtration $(\mathcal{F}_t)_{t\geq 0}$ that satisfies the usual conditions. Denote by $\mathbb{E}$ the probability expectation with respect to (w.r.t.) $\mathbb{P}$. Let $|\cdot|$ and $\|\cdot\|$ represent the Euclidean norm for the vector in $\mathbb{R}^d$ and the trace norm for the matrix in $\mathbb{R}^d \otimes \mathbb{R}^m$, respectively. For $p \geq 1$, the set of random variables Z in $\mathbb{R}^d$ with $\mathbb{E}|Z|^p < \infty$ is denoted by $L^p(\mathbb{R}^d)$. Let $c_1 \wedge c_2 = \min\{c_1, c_2\}$ and $c_1 \vee c_2 = \max\{c_1, c_2\}$ for real numbers c_1, c_2. Let $\mathcal{P}(\mathbb{R}^d)$ be the collection of all probability measures on $\mathbb{R}^d$. For $p \geq 1$, define

$$\mathcal{P}_p(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \left(\int_{\mathbb{R}^d} |x|^p \mu(dx) \right)^{1/p} < \infty \right\}.$$

Denote by $\delta_x(\cdot)$ the Dirac measure at point $x \in \mathbb{R}^d$, which belongs to $\mathcal{P}_p(\mathbb{R}^d)$. For $p \geq 1$, the Wasserstein distance between $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$ is defined by

$$\mathbb{W}_p(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu,\nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(dx, dy) \right)^{1/p},$$

where $\mathcal{C}(\mu, \nu)$ is the family of all couplings for μ, ν, i.e., $\pi(\cdot, \mathbb{R}^d) = \mu(\cdot)$ and $\pi(\mathbb{R}^d, \cdot) = \nu(\cdot)$.

Background

Consider the following general (stochastic McKean-Vlasov equation) SMVE of the form

$$dX_t = b(X_t, \mathcal{L}_{X_t})dt + \sigma(X_t, \mathcal{L}_{X_t})dW_t, \quad t \geq 0, \tag{1}$$

with the initial vale $X_0 \in L^{\widetilde{p}}_{\mathcal{F}_0}$ Here, $\mathcal{L}_{X_t}$ is the law of X at time t, and $b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^m$ are both Borel-measurable. And W(t) is an m-dimensional Brownian motion on the given probability space.

$$\begin{cases} \mathrm{d}y(t) = \left(2y(t) + \int_{\mathbb{R}} z\mu_t(dz)\right)\mathrm{d}t + y(t)\mathrm{d}B(t), \\ y(0) = y_0. \end{cases} \tag{2}$$

The interacting and non-interacting particle systems

For any $j \in \mathbb{S}_N := \{1, 2, \cdots, N\}$, let (W^j, X_0^j) be independent copies of (W, X_0). Then, the non-interacting particle system w.r.t. (1) is introduced by

$$dX_t^j = b(X_t^j, \mu_t^{X_j})dt + \sigma(X_t^j, \mu_t^{X_j})dW_t^j, \quad j \in \mathbb{S}_N, \tag{3}$$

with the initial value X_0^j, where $\mu_t^{X_j}$ is the law of X_j at t. Here, $\mu_t^{X_j} = \mu_t^X$.
To approximate the law, the corresponding interacting particles system is defined by

$$dX_t^{j,N} = b(X_t^{j,N}, \mu_t^{X,N})dt + \sigma(X_t^{j,N}, \mu_t^{X,N})dW_t^j, \quad j \in \mathbb{S}_N, \tag{4}$$

with the initial value X_0^j, where the empirical measure is defined by $\mu_t^{X,N}(\cdot) = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}(\cdot)$.

Theories of SMVEs

The work on SMVEs was initiated by McKean, who was inspired by Kac's Programme in Kinetic Theory. Sznitman investigated the existence and uniqueness of the results under a global Lipschitz condition.

- H. McKean, *A class of Markov processes associated with nonlinear parabolic equations,* Proc. Nat. Acad. Sci., 56 (1967), 1907-1911.
- A. Sznitman, *Topics in propagation of chaos,* Springer, 1991.
- M. Kac, *Foundations of Kinetic Theory. the Third Berkeley Symposium on Mathematical Statistics and Probability, III(1954-1955), 171-197,* University of California Press, Berkeley and Los Angeles.

- Existence and uniqueness theory: Sznitman 1991, Wang 2018, Dos Reis-Salkeld-Tugaut 2019, Kumar- Neelima-Reisinger-Stockinger 2022,
- Harnack inequality: Wang 2018, Huang-Wang 2019,
- Feyman-Kac formula: Buckdahn-Li-Peng-Rainer 2017, Crisan-McMurray 2018, Ren-Röckner-Wang 2022,
- Gradient estimation: Song 2020,
- Maximum likelihood estimation: Wen-Wang-Mao-Xiao 2016

Numerical schemes for SMVEs

By means of the stochastic particle method, many scholars have discussed the convergence of the numerical schemes, such as

- tamed EM scheme: Dos Reis-Engelhardt-Smith 2022, Neelima-Biswas-Kumar-Dos Reis-Reisinger 2020,
- split-step scheme: Chen-Dos Reis 2022,
- multi-level Monte-Carlo scheme: Bao-Reisinger-Ren- Stockinger 2023,
- implicit EM scheme: Dos Reis-Engelhardt-Smith 2022;
- tamed Milstein scheme: Bao-Reisinger-Ren- Stockinger 2021, 2023.
- adaptive EM and Milstein schemes: Reisinger-Stockinger 2022
- ....

The stability of the classical SDEs

The moment stability and almost sure stability have been systematically investigated in (Mao 1994, Khasminskii 2012). And these two types of stability of the numerical schemes for classical SDEs have been widely discussed, such as

- EM scheme: Higham-Mao-Yuan 2007, Pang-Deng-Mao 2008,
- θ-EM scheme: C. Huang 2012, X. Mao 2015, Wang-Gan 2010, Zong-Wu 2014,
- truncated EM scheme: Geng-Song-Lu-Liu 2021, Lan-Xia-Wang 2019,
- backward EM scheme: Guo-Li 2018,
- split-step EM scheme: Xie-Zhang 2020.

The stability of the solutions to SMVEs

- The exponential stability of the semilinear McKean-Vlasov stochastic evolution equation was shown in (Govindan-Ahmed 2015).
- The moment stability and almost sure stability of SMVEs (Ding-Qiao 2021) and multivalued SMVEs (Gong-Qiao 2021) were presented.
- The stabilization of SMVEs with feedback control via discrete time state observation was revealed in (Wu-Hu-Gao-Yuan 2022), where the stability of the corresponding interacting particle system was also analyzed due to the unobservability of the law of SMVEs.

Assumptions

Assumption 1

For any $R > 0$, there exist constants $K_R > 0$ and $K_0 >$ such that

$$|b(x, \mu) - b(y, \nu)|^2 \vee \|\sigma(x, \mu) - \sigma(y, \nu)\|^2 \leq K_R|x - y|^2 + K_0\mathbb{W}_2(\mu, \nu)^2,$$

for any $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq R$.

Assumption 2

There exist positive constants K_1, K_2 such that

$$2\langle x-y, b(x, \mu)-b(y, \nu)\rangle+\|\sigma(x, \mu)-\sigma(y, \nu)\|^2 \leq -K_1|x-y|^2+K_2\mathbb{W}_2(\mu, \nu)^2,$$

for any $x, y \in \mathbb{R}^d$ and $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$.

Assumption 3

There exist three constants $a_1, a_2 > 0$ and $q > 2$ such that

$$2\langle x, b(x,\mu)\rangle + (q-1)\|\sigma(x,\mu)\|^2 \le -a_1|x|^2 + a_2\mathcal{W}_2(\mu)^2,$$

for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

Under these assumptions, the SMVE (1) admits a unique strong solution (Kumar-Neelima-Reisinger-Stockinger 2022).

The interacting and non-interacting particle systems

For any $j \in \mathbb{S}_N := \{1, 2, \cdots, N\}$, let (W^j, X_0^j) be independent copies of (W, X_0). Then, the non-interacting particle system w.r.t. (1) is introduced by

$$dX_t^j = b(X_t^j, \mu_t^{X_j})dt + \sigma(X_t^j, \mu_t^{X_j})dW_t^j, \quad j \in \mathbb{S}_N, \tag{5}$$

with the initial value X_0^j, where $\mu_t^{X_j}$ is the law of X_j at t. Here, $\mu_t^{X_j} = \mu_t^X$.
To approximate the law, the corresponding interacting particles system is defined by

$$dX_t^{j,N} = b(X_t^{j,N}, \mu_t^{X,N})dt + \sigma(X_t^{j,N}, \mu_t^{X,N})dW_t^j, \quad j \in \mathbb{S}_N, \tag{6}$$

with the initial value X_0^j, where the empirical measure is defined by $\mu_t^{X,N}(\cdot) = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}(\cdot)$.

Stability of interacting particle system

Theorem 1

Let Assumptions 1-3 hold. If $a_1 > a_2$, then the solution $X_t^{j,N}$ to interacting particle system (6) is exponentially stable in mean-square sense, i.e., for any $j \in \mathbb{S}_N$ and some $a \in (0, a_1 - a_2)$,

$$\lim_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|X_t^{j,N}|^2) \leq -a.$$

Since the particles are all identically distributed for any $j \in \mathbb{S}_N$, we derive $\mathbb{E}[\frac{1}{N}\sum_{i=1}^N |X_t^{i,N}|^2] = \mathbb{E}[|X_t^{j,N}|^2]$. By taking Assumption 3 into consideration, we have

$$\begin{aligned} e^{at}\mathbb{E}|X_t^{j,N}|^2 \leq & \mathbb{E}|X_0^j|^2 + \int_0^t ((a-a_1)\mathbb{E}|X_s^{j,N}|^2 + a_2\mathbb{E}[\frac{1}{N}\sum_{i=1}^N |X_s^{i,N}|^2])ds \\ \leq & \mathbb{E}|X_0^j|^2 + \int_0^t (a_2 - a_1 + a)\mathbb{E}|X_s^{j,N}|^2 ds, \end{aligned}$$

where we use a result of Wasserstein metric $\mathbb{W}_2(\mu_t^{X,N}, \delta_0)^2 \leq \frac{1}{N}\sum_{i=1}^N |X_t^{i,N}|^2$.

Theorem 2

Let Assumptions 1-3 hold. If $a_1 > a_2$, then the solution $X_t^{j,N}$ to interacting particle system (6) is almost surely stable, i.e., for some $a \in (0, a_1 - a_2)$,

$$\lim_{t\to\infty} \frac{1}{t} \log \left(\frac{1}{N} \sum_{i=1}^{N} |X_t^{i,N}|^2 \right) \leq -a \quad a.s.$$

summing from $j = 1$ to N and then dividing it by N leads to

$$\begin{aligned} e^{at}\frac{1}{N}\sum_{i=1}^{N}|X_t^{i,N}|^2 =& \frac{1}{N}\sum_{i=1}^{N}|X_0^i|^2 + \int_0^t e^{as}\left((a - a_1 + a_2)\frac{1}{N}\sum_{i=1}^{N}|X_s^{i,N}|^2\right)ds \\ &+ 2\int_0^t e^{as}\frac{1}{N}\sum_{i=1}^{N}\langle X_s^{i,N}, \sigma(X_s^{i,N}, \mu_s^{X,N})\rangle dW_s^i \\ \leq& \frac{1}{N}\sum_{i=1}^{N}|X_0^i|^2 + 2\int_0^t e^{as}\frac{1}{N}\sum_{i=1}^{N}\langle X_s^{i,N}, \sigma(X_s^{i,N}, \mu_s^{X,N})\rangle dW_s^i, \end{aligned}$$

where Assumption 3 is used.

Propagation of chaos in infinite horizon

Theorem 3

Let Assumptions 1-3 hold. If $2K_2 < K_1$ and $a_2 < a_1$, then for some $a \in (0, a_1 - a_2)$,

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}|X_t^i - X_t^{i,N}|^2 \leq \frac{2K_2 G_{d,q}(\mathbb{E}|X_0|^q)^{\frac{2}{q}} e^{-at}\Phi(N)}{K_1 - 2K_2}, \tag{7}$$

where $\Phi(N) := \begin{cases} N^{-\frac{1}{2}} + N^{-\frac{q-2}{q}}, & d < 4 \text{ and } q \neq 4, \\ N^{-\frac{1}{2}}\log(1+N) + N^{-\frac{q-2}{q}}, & d = 4 \text{ and } q \neq 4, \\ N^{-\frac{2}{d}} + N^{-\frac{q-2}{q}}, & d > 4 \text{ and } q \neq \frac{d}{d-2}. \end{cases}$

Obviously,

$$\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}|X_t^i - X_t^{i,N}|^2 = 0.$$

Proof of Theorem 3

By Itô's formula, we derive from (5) and (6) that

$$\begin{aligned} |X_t^j - X_t^{j,N}|^2 =& 2\int_0^t \langle X_s^j - X_s^{j,N}, b(X_s^j, \mu_s^{X^j}) - b(X_s^{j,N}, \mu_s^{X,N})\rangle ds \\ &+ \int_0^t \|\sigma(X_s^j, \mu_s^{X^j}) - \sigma(X_s^{j,N}, \mu_s^{X,N})\|^2 ds \\ &+ 2\int_0^t \langle X_s^j - X_s^{j,N}, \sigma(X_s^j, \mu_s^{X^j}) - \sigma(X_s^{j,N}, \mu_s^{X,N})\rangle dW_s^j. \end{aligned} \tag{8}$$

Taking the expectation on both sides of (8) and the derivative w.r.t. t yield that

$$\frac{d}{dt}\mathbb{E}|X_t^j - X_t^{j,N}|^2 = 2\mathbb{E}\langle X_t^j - X_t^{j,N}, b(X_t^j, \mu_t^{X^j}) - b(X_t^{j,N}, \mu_t^{X,N})\rangle + \mathbb{E}\|\sigma(X_t^j, \mu_t^{X^j}) - \sigma(X_t^{j,N}, \mu_t^{X,N})\|^2.$$

Then, we take into account Assumption 2 to arrive at

$$\frac{d}{dt}\mathbb{E}|X_t^j - X_t^{j,N}|^2 \le -K_1\mathbb{E}|X_t^j - X_t^{j,N}|^2 + K_2\mathbb{E}\big[\mathbb{W}_2(\mu_t^{X^j}, \mu_t^{X,N})^2\big]. \tag{9}$$

By summing (9) from $j = 1$ to N and then dividing it by N, one can see that

$$\begin{aligned}\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}|X_t^i - X_t^{i,N}|^2 \leq & -\frac{K_1}{N}\sum_{i=1}^{N}\mathbb{E}|X_t^i - X_t^{i,N}|^2 + \frac{2K_2}{N}\sum_{i=1}^{N}\mathbb{E}|X_t^i - X_t^{i,N}|^2 + 2K_2\mathbb{E}[\mathbb{W}_2(\mu_t^{X_j}, \mu_t^N)^2] \\ \leq & (2K_2 - K_1)\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}|X_t^i - X_t^{i,N}|^2 + 2K_2\mathbb{E}[\mathbb{W}_2(\mu_t^{X_j}, \mu_t^N)^2],\end{aligned} \tag{10}$$

where we use the fact that

$$\mathbb{E}[\mathbb{W}_2(\mu_t^{X_j}, \mu_t^{X,N})^2] \leq 2\mathbb{E}[\mathbb{W}_2(\mu_t^{X_j}, \mu_t^N)^2] + 2\mathbb{E}[\mathbb{W}_2(\mu_t^N, \mu_t^{X,N})^2],$$

with $\mu_t^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i}$.

By choosing $p = 2$ in Theorem 1 of [Theorem 1 in Fournier-Guillin 2015], we get that, for $q > 2$ and $t \geq 0$,

$$\mathbb{E}\big[\mathbb{W}_2(\mu_t^{X_j}, \mu_t^N)^2\big] \leq G_{d,q}(\mathbb{E}|X_t|^q)^{\frac{2}{q}}\Phi(N), \tag{11}$$

where $G_{d,q}$ is a constant which only depends on d, q, and

$$\Phi(N) := \begin{cases} N^{-\frac{1}{2}} + N^{-\frac{q-2}{q}}, & d < 4 \text{ and } q \neq 4, \\ N^{-\frac{1}{2}}\log(1+N) + N^{-\frac{q-2}{q}}, & d = 4 \text{ and } q \neq 4, \\ N^{-\frac{2}{d}} + N^{-\frac{q-2}{q}}, & d > 4 \text{ and } q \neq \frac{d}{d-2}. \end{cases}$$

Actually, the Itô formula with Assumption 3 leads to

$$(\mathbb{E}|X_t|^q)^{\frac{2}{q}} \leq e^{-at}(\mathbb{E}|X_0|^q)^{\frac{2}{q}}. \tag{12}$$

Therefore, inserting (11) and (12) into (10) yields that

$$\frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}|X_t^i - X_t^{i,N}|^2 \leq (2K_2 - K_1)\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}|X_t^i - X_t^{i,N}|^2 + 2K_2G_{d,q}(\mathbb{E}|X_0|^q)^{\frac{2}{q}}e^{-at}\Phi(N). \tag{13}$$

Denote

$$\varrho(t) = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}|X_t^i - X_t^{i,N}|^2.$$

From (13), it holds that

$$\varrho'(t) \leq (2K_2 - K_1)\varrho(t) + 2K_2G_{d,q}(\mathbb{E}|X_0|^q)^{\frac{2}{q}}e^{-at}\Phi(N),$$

with $\varrho(0) = 0$. By integrating this Gronwall-like differential inequality, we draw the conclusion.

Theorem 4

Let all conditions in Theorem 3 hold. Then

$$\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}|X_t^i - X_t^{i,N}|^2 = 0 \quad a.s.$$

Proof: Firstly, let $\epsilon \in (0, a)$ be arbitrary. By the technique of Chebyshev's inequality and (7), we have

$$\begin{aligned}\mathbb{P}\Big(\frac{1}{N}\sum_{i=1}^{N}|X_t^i - X_t^{i,N}|^2 > e^{-(a-\epsilon)t}\Phi(N)\Big) &\leq \frac{\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}|X_t^i - X_t^{i,N}|^2}{e^{-(a-\epsilon)t}\Phi(N)} \\ &\leq \frac{2K_2 G_{d,q}(\mathbb{E}|X_0|^q)^{\frac{2}{q}} e^{-\varepsilon t}}{(K_1 - 2K_2)},\end{aligned}$$

where $\Phi(N)$ is defined in Theorem 3. The Borel-Cantelli lemma allows us to know that, for almost all $\omega \in \Omega$,

$$\frac{1}{N}\sum_{i=1}^{N}|X_t^i - X_t^{i,N}|^2 \leq e^{-(a-\epsilon)t}\Phi(N) \tag{14}$$

holds for all but finite t. Thus, there exists a $T^*(\omega), \forall\omega \in \Omega$ such that (14) holds whenever $t \geq T^*$, which means

$$\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}|X_t^i - X_t^{i,N}|^2 = 0 \quad a.s.$$

Stabilities of EM scheme for interacting particle system

After constructing the conventional EM scheme for interacting particle system, we provide the mean-square and almost sure stabilities of the numerical solution. Suppose that there exists a positive integer M such that $\Delta = \frac{1}{M}$. For each $j \in \mathbb{S}_N$ and the given time-step Δ, the EM scheme of (6) in discretization version is:

$$Y_{t_{k+1}}^{j,N} = Y_{t_k}^{j,N} + b(Y_{t_k}^{j,N}, \mu_{t_k}^{Y,N})\Delta + \sigma(Y_{t_k}^{j,N}, \mu_{t_k}^{Y,N})\Delta W_{t_k}^{j}, \tag{15}$$

where $\Delta W_{t_k}^{j} = W_{t_{k+1}}^{j} - W_{t_k}^{j}$ and $\mu_{t_k}^{Y,N}(\cdot) = \frac{1}{N}\sum_{i=1}^{N} \delta_{Y_{t_k}^{i,N}}(\cdot)$.

Assumption 4

There exist two constants $b_1, b_2 > 0$ such that

$$|b(x,\mu)|^2 \leq b_1|x|^2 + b_2\mathcal{W}_2(\mu)^2,$$

for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

Theorem 5

Let Assumptions 1-4 hold with $0 < a_2 < a_1$. Then for any $j \in \mathbb{S}_N$, there exists a stepsize $\Delta_0 \in (0,1)$ such that for any $\Delta \in (0, \Delta_0)$, the numerical solution $Y_{t_k}^{j,N}$ is exponentially stable in mean-square sense, i.e.,

$$\lim_{k\to\infty} \frac{1}{k\Delta} \log\left(\mathbb{E}|Y_{t_k}^{j,N}|^2\right) \le -\theta_\Delta^*,$$

where θ_Δ^* is a positive constant satisfying

$$\lim_{\Delta\to 0} \theta_\Delta^* = a_1 - a_2.$$

Here, the constraint on Δ_0 is given in the proof.

Proof of Theorem 5

By Assumptions 3 and 4, it is easy to see that

$$\begin{aligned}\mathbb{E}|Y^{j,N}_{t_{k+1}}|^2 \le& \mathbb{E}|Y^{j,N}_{t_k}|^2 + b_1\Delta^2\mathbb{E}|Y^{j,N}_{t_k}|^2 + \frac{b_2}{N}\Delta^2\sum_{i=1}^{N}\mathbb{E}|Y^{i,N}_{t_k}|^2 - a_1\mathbb{E}|Y^{j,N}_{t_k}|^2\Delta + \frac{a_2}{N}\Delta\sum_{i=1}^{N}\mathbb{E}|Y^{i,N}_{t_k}|^2 \\ \le& \mathbb{E}|Y^{j,N}_{t_k}|^2 + (b_1+b_2)\Delta^2\mathbb{E}|Y^{j,N}_{t_k}|^2 + (a_2-a_1)\Delta\mathbb{E}|Y^{j,N}_{t_k}|^2.\end{aligned} \quad (16)$$

For any constant $\lambda > 1$, we get that

$$\lambda^{(k+1)\Delta}|Y^{j,N}_{t_{k+1}}|^2 - \lambda^{k\Delta}|Y^{j,N}_{t_k}|^2 = \lambda^{(k+1)\Delta}\left(|Y^{j,N}_{t_{k+1}}|^2 - |Y^{j,N}_{t_k}|^2\right) + \left(\lambda^{(k+1)\Delta} - \lambda^{k\Delta}\right)|Y^{j,N}_{t_k}|^2. \quad (17)$$

Taking (16) and (17) into consideration leads to

$$\lambda^{(k+1)\Delta}\mathbb{E}|Y^{j,N}_{t_{k+1}}|^2 - \lambda^{k\Delta}\mathbb{E}|Y^{j,N}_{t_k}|^2 \le \lambda^{(k+1)\Delta}\left[(b_1+b_2)\Delta^2 + (a_2-a_1)\Delta + 1 - \lambda^{-\Delta}\right]\mathbb{E}|Y^{j,N}_{t_k}|^2,$$

which yields

$$\lambda^{k\Delta}\mathbb{E}|Y^{j,N}_{t_k}|^2 - \mathbb{E}|Y^{j,N}_{0}|^2 \le \sum_{l=0}^{k-1}\lambda^{(l+1)\Delta}\left[(b_1+b_2)\Delta^2 + (a_2-a_1)\Delta + 1 - \lambda^{-\Delta}\right]\mathbb{E}|Y^{j,N}_{t_l}|^2.$$

Define

$$h(\lambda) = \lambda^{\Delta}\left((b_1+b_2)\Delta^2 + (a_2-a_1)\Delta + 1\right) - 1.$$

For any $\Delta \in (0, \Delta_0)$ with $\Delta_0 = \Delta_1 \wedge \Delta_2 \wedge 1$, there exists a unique $\lambda_\Delta^* > 1$ such that $h(\lambda_\Delta^*) = 0$, where λ_Δ^* depends on the stepsize Δ. Letting $\lambda = \lambda_\Delta^*$ gives that

$$\lambda_\Delta^{*\,k\Delta}\mathbb{E}|Y_{t_k}^{j,N}|^2 \leq \mathbb{E}|Y_0^{j,N}|^2.$$

By choosing $\theta_\Delta^* > 0$ such that $\lambda_\Delta^* = \exp(\theta_\Delta^*)$, we can get that

$$\mathbb{E}|Y_{t_k}^{j,N}|^2 \leq \exp(-k\Delta\theta_\Delta^*)\mathbb{E}|Y_0^{j,N}|^2.$$

Define

$$\widetilde{h}(\lambda) = \frac{h(\lambda)}{\Delta\lambda^\Delta} = (b_1 + b_2)\Delta + a_2 - a_1 + \frac{(1 - \lambda^{-\Delta})}{\Delta},$$

with $\widetilde{h}(\lambda_\Delta^*) = h(\lambda_\Delta^*) = 0$.

It is worth noting that $\lim_{\Delta \to 0} \frac{(1 - \lambda_\Delta^{*\,-\Delta})}{\Delta} = \lim_{\Delta \to 0} \theta_\Delta^*$. Thus,

$$\lim_{\Delta \to 0} \widetilde{h}(\lambda_\Delta^*) = \lim_{\Delta \to 0} \theta_\Delta^* + a_2 - a_1 = 0.$$

The desired result can be proved.

Assumption 5

There exist constants $c_1, c_2 > 0$ such that

$$2\langle x, b(x,\mu)\rangle + m\|\sigma(x,\mu)\|^2 \leq -c_1|x|^2 + c_2\mathcal{W}_2(\mu)^2,$$

for any $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, where m is the dimension of Brownian motion.

Theorem 6

Let Assumptions 1, 2, 4 and 5 hold with $0 < c_2 < c_1$. Then there exists a stepsize $\overline{\Delta}_0 \in (0,1)$ such that for any $\Delta \in (0, \overline{\Delta}_0)$, the numerical solution $Y^{j,N}_{t_k}$ is almost surely exponentially stable, i.e.,

$$\lim_{k\to\infty} \frac{1}{k\Delta} \log\Big(\frac{1}{N}\sum_{i=1}^{N} |Y^{i,N}_{t_k}|^2\Big) \le -\xi^*_\Delta,$$

where ξ^*_Δ is a positive constant satisfying

$$\lim_{\Delta\to 0} \xi^*_\Delta = c_1 - c_2.$$

Here, we provide the constraint on $\overline{\Delta}_0$ in the proof.

Proof of Theorem 6

By Assumptions 4 and 5, we obtain

$$\begin{aligned}|Y^{j,N}_{t_{k+1}}|^2 \le & |Y^{j,N}_{t_k}|^2 + \Delta\Big(- c_1|Y^{j,N}_{t_k}|^2 + \frac{c_2}{N}\sum_{i=1}^{N}|Y^{i,N}_{t_k}|^2 + b_1\Delta|Y^{j,N}_{t_k}|^2 + \frac{b_2\Delta}{N}\sum_{i=1}^{N}|Y^{i,N}_{t_k}|^2\Big) + \gamma_k^j \\ \le & |Y^{j,N}_{t_k}|^2 + \Delta(b_1\Delta - c_1)|Y^{j,N}_{t_k}|^2 + \Delta(b_2\Delta + c_2)\frac{1}{N}\sum_{i=1}^{N}|Y^{i,N}_{t_k}|^2 + \gamma_k^j,\end{aligned} \tag{18}$$

where

$$\begin{aligned}\gamma_k^j = & \|\sigma(Y^{j,N}_{t_k}, \mu^{Y,N}_{t_k})\|^2|\Delta W^j_{t_k}|^2 - m\|\sigma(Y^{j,N}_{t_k}, \mu^{Y,N}_{t_k})\|^2\Delta \\ & + 2\langle Y^{j,N}_{t_k}, \sigma(Y^{j,N}_{t_k}, \mu^{Y,N}_{t_k})\Delta W^j_{t_k}\rangle + 2\langle b(Y^{j,N}_{t_k}, \mu^{Y,N}_{t_k})\Delta, \sigma(Y^{j,N}_{t_k}, \mu^{Y,N}_{t_k})\Delta W^j_{t_k}\rangle.\end{aligned}$$

Combining (17) and (18) gives that

$$\begin{aligned}\lambda^{k\Delta}|Y^{j,N}_{t_k}|^2 - |Y^{j,N}_0|^2 \le \sum_{l=0}^{k-1}\lambda^{(l+1)\Delta}\big[& (1 + b_1\Delta^2 - c_1\Delta - \lambda^{-\Delta})|Y^{j,N}_{t_l}|^2 \\ & + (b_2\Delta^2 + c_2\Delta)\frac{1}{N}\sum_{i=1}^{N}|Y^{i,N}_{t_l}|^2 + \gamma_l^j\big].\end{aligned}$$

Define

$$f(\lambda) = \lambda^{\Delta}(1 + b_1\Delta^2 - c_1\Delta) - 1. \tag{19}$$

For any $\Delta < \overline{\Delta}_1 \wedge \overline{\Delta}_2 \wedge 1$, there exists a unique $\vartheta^*_\Delta > 1$ such that $f(\vartheta^*_\Delta) = 0$, where ϑ^*_Δ depends on the stepsize Δ. By taking $\lambda = \vartheta^*_\Delta$, we have

$$ {\vartheta^*_\Delta}^{k\Delta} |Y^{j,N}_{t_k}|^2 \leq |Y^{j,N}_0|^2 + \frac{1}{N}(b_2\Delta^2 + c_2\Delta) \sum_{l=0}^{k-1} \sum_{i=1}^{N} {\vartheta^*_\Delta}^{(l+1)\Delta} |Y^{i,N}_{t_l}|^2 + \sum_{l=0}^{k-1} {\vartheta^*_\Delta}^{(l+1)\Delta} \gamma^j_l . \tag{20} $$

Every particle satisfies (20). Summing N inequalities gives the result

$$ \sum_{i=1}^{N} {\vartheta^*_\Delta}^{k\Delta} |Y^{i,N}_{t_k}|^2 \leq \sum_{i=1}^{N} |Y^{i,N}_0|^2 + (b_2\Delta^2 + c_2\Delta) \sum_{l=0}^{k-1} \sum_{i=1}^{N} {\vartheta^*_\Delta}^{(l+1)\Delta} |Y^{i,N}_{t_l}|^2 + \sum_{i=1}^{N} \sum_{l=0}^{k-1} {\vartheta^*_\Delta}^{(l+1)\Delta} \gamma^i_l . $$

Let $A_l = \sum_{i=1}^{N} {\vartheta^*_\Delta}^{l\Delta} |Y^{i,N}_{t_l}|^2$, $B_l = \sum_{i=1}^{N} {\vartheta^*_\Delta}^{(l+1)\Delta} \gamma^i_l$, $l \in \{0, 1, \cdots, k-1\}$, and

$$ X_k = \sum_{i=1}^{N} |Y^{i,N}_0|^2 + {\vartheta^*_\Delta}^{\Delta} (b_2\Delta^2 + c_2\Delta) \sum_{l=0}^{k-1} A_l + \sum_{l=0}^{k-1} B_l . $$

We construct

$$ X_k - X_{k-1} = {\vartheta^*_\Delta}^{\Delta} (b_2\Delta^2 + c_2\Delta) A_{k-1} + B_{k-1} . \tag{21} $$

Since $A_l \leq X_l$ for $l \in \{0, 1, \cdots, k\}$, (21) becomes into

$$\begin{aligned} X_k =& X_{k-1} + {\vartheta^*_\Delta}^\Delta (b_2\Delta^2 + c_2\Delta) A_{k-1} + B_{k-1} \\ \leq& \big(1 + {\vartheta^*_\Delta}^\Delta (b_2\Delta^2 + c_2\Delta)\big) X_{k-1} + B_{k-1}. \end{aligned} \tag{22}$$

Owing to (22), it can be obtained by iteration that

$$X_k \leq \big(1 + {\vartheta^*_\Delta}^\Delta (b_2\Delta^2 + c_2\Delta)\big)^{k-1} X_1 + \sum_{l=0}^{k-1} \big(1 + {\vartheta^*_\Delta}^\Delta (b_2\Delta^2 + c_2\Delta)\big)^{k-1-l} B_l.$$

Thus,

$$\sum_{i=1}^N {\vartheta^*_\Delta}^{k\Delta} |Y^{i,N}_{t_k}|^2 \leq \exp\big(k\Delta {\vartheta^*_\Delta}^\Delta (b_2\Delta + c_2)\big)\Big(\sum_{i=1}^N |Y^{i,N}_0|^2 + \sum_{l=1}^{k-1} B_l\Big), \tag{23}$$

which follows from an elementary inequality $1 + z \leq e^z$ for $z \in \mathbb{R}$.
Then, dividing both sides of (23) by $N \exp\big(k\Delta {\vartheta^*_\Delta}^\Delta (b_2\Delta + c_2)\big)$ yields that

$$\Big(\frac{{\vartheta^*_\Delta}^{k\Delta}}{\exp\big(k\Delta {\vartheta^*_\Delta}^\Delta (b_2\Delta + c_2)\big)}\Big)\frac{1}{N}\sum_{i=1}^N |Y^{i,N}_{t_k}|^2 \leq \frac{1}{N}\sum_{i=1}^N |Y^{i,N}_0|^2 + \frac{1}{N}\sum_{l=1}^{k-1} B_l. \tag{24}$$

Define

$$Z_k = \frac{1}{N}\sum_{i=1}^{N}|Y_0^{i,N}|^2 + M_k,$$

where $M_k = \frac{1}{N}\sum_{l=1}^{k-1} B_l = \frac{1}{N}\sum_{l=1}^{k-1}\sum_{i=1}^{N} \vartheta_\Delta^{*\ (l+1)\Delta}\gamma_l^i$.
For any $j \in \mathbb{S}_N$, it is worth noting that $\mathbb{E}\big[(|\Delta W_{t_{k-1}}^j|^2 - m\Delta)|\mathcal{F}_{(k-1)\Delta}\big] = 0$, so

$$\begin{aligned}
\mathbb{E}\big[\gamma_{k-1}^j|\mathcal{F}_{(k-1)\Delta}\big] =& \|\sigma(Y_{t_{k-1}}^{j,N}, \mu_{t_{k-1}}^{Y,N})\|^2 \mathbb{E}\big[(|\Delta W_{t_{k-1}}^j|^2 - m\Delta)|\mathcal{F}_{(k-1)\Delta}\big] \\
&+ 2\mathbb{E}\big[\langle Y_{t_k}^{j,N}, \sigma(Y_{t_k}^{j,N}, \mu_{t_k}^{Y,N})\Delta W_{t_k}^j\rangle|\mathcal{F}_{(k-1)\Delta}\big] \\
&+ 2\mathbb{E}\big[\langle b(Y_{t_k}^{j,N}, \mu_{t_k}^{Y,N})\Delta, \sigma(Y_{t_k}^{j,N}, \mu_{t_k}^{Y,N})\Delta W_{t_k}^j\rangle|\mathcal{F}_{(k-1)\Delta}\big] \\
=& 0.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E}\big[M_k|\mathcal{F}_{(k-1)\Delta}\big] =& \mathbb{E}\big[M_{k-1} + \frac{1}{N}\sum_{i=1}^{N}\gamma_{k-1}^i|\mathcal{F}_{(k-1)\Delta}\big] \\
=& M_{k-1} + \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\big[\gamma_{k-1}^i|\mathcal{F}_{(k-1)\Delta}\big] \\
=& M_{k-1}.
\end{aligned}$$

Therefore, applying Lemma 2.2 in [Mao 1997] with (24) gives that for any $j \in \mathbb{S}_N$ and $\Delta < \overline{\Delta}_1 \wedge \overline{\Delta}_2 \wedge 1$,

$$\lim_{k\to\infty} \eta_k^* \frac{1}{N} \sum_{i=1}^{N} |Y_{t_k}^{i,N}|^2 \leq \lim_{k\to\infty} Z_k < \infty \quad a.s. \tag{25}$$

where $\eta_k^* = {\vartheta_\Delta^*}^{k\Delta} / \exp\left(k\Delta {\vartheta_\Delta^*}^{\Delta}(b_2\Delta + c_2)\right)$. Choose τ_Δ^* and ξ_Δ^* such that $\vartheta_\Delta^* = \exp(\tau_\Delta^*)$ and $\eta_k^* = \exp(k\Delta\xi_\Delta^*)$, then (25) reads

$$\lim_{k\to\infty} \exp(k\Delta\xi_\Delta^*) \frac{1}{N} \sum_{i=1}^{N} |Y_{t_k}^{i,N}|^2 < \infty.$$

Define

$$\widetilde{f}(\lambda) = b_1\Delta - c_1 + \frac{(1-\lambda^{-\Delta})}{\Delta}.$$

According to $f(\vartheta_\Delta^*) = 0$, we obviously have

$$\widetilde{f}(\vartheta_\Delta^*) = b_1\Delta - c_1 + \frac{(1-{\vartheta_\Delta^*}^{-\Delta})}{\Delta} = b_1\Delta - c_1 + \frac{(1-\exp(-\Delta\tau_\Delta^*))}{\Delta} = 0.$$

Thus,

$$\lim_{\Delta\to 0} \tau_\Delta^* = c_1.$$

From (19) and the chosen τ^*_Δ and ξ^*_Δ, we therefore have that

$$\xi^*_\Delta = \tau^*_\Delta - {\vartheta^*_\Delta}^\Delta (b_2\Delta + c_2) = -\frac{\log(1 + b_1\Delta^2 - c_1\Delta)}{\Delta} - \frac{b_2\Delta + c_2}{1 + b_1\Delta^2 - c_1\Delta}.$$

Define

$$g(\Delta) = \log(1 + b_1\Delta^2 - c_1\Delta) + \frac{b_2\Delta^2 + c_2\Delta}{1 + b_1\Delta^2 - c_1\Delta}.$$

Then,

$$g'(\Delta) = \frac{2{b_1}^2\Delta^3 - (b_1c_2 + c_1b_2 + 3b_1c_1)\Delta^2 + (2b_1 + 2b_2 + {c_1}^2)\Delta + c_2 - c_1}{(1 + b_1\Delta^2 - c_1\Delta)^2}.$$

Thus, there exists a $\overline{\Delta}_3$ such that $g'(\Delta) < 0$ for $\Delta \in (0, \overline{\Delta}_3)$. Let $\overline{\Delta}_0 = \overline{\Delta}_1 \wedge \overline{\Delta}_2 \wedge \overline{\Delta}_3 \wedge 1$. Since $g(0) = 0$ and $g'(\Delta) < 0$ for $\Delta \in (0, \overline{\Delta}_0)$, we have $g(\Delta) < 0$ for $\Delta \in (0, \overline{\Delta}_0)$, which implies $\xi^*_\Delta > 0$.
It follows that

$$\lim_{\Delta\to 0} \xi^*_\Delta = \lim_{\Delta\to 0} \tau^*_\Delta - \lim_{\Delta\to 0} \left({\vartheta^*_\Delta}^\Delta (b_2\Delta + c_2)\right) = c_1 - c_2.$$

Then we derive the desired statement.

Numerical examples

Consider the following scalar SMVE:

$$dX_t = \left(-\frac{7}{2}X_t + \mathbb{E}X_t\right)dt + X_t dW_t, \tag{26}$$

with random initial data $X_0 \sim \mathcal{N}(2,1)$, where $\mathcal{N}(\cdot,\cdot)$ is normal distribution.

By using 1000 particles and 100 trajectories, we find out that the rate of stability in mean-square sense is not greatly affected by the number of particles.

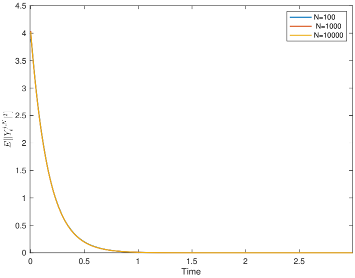

Figure: mean-square stability with stepsize $\Delta = 0.01$

By Theorem 5, we observe that the rate of stability is related to the stepsize. When the stepsize is smaller, the rate is larger, and the rate of mean-square stability is $a_2 - a_1$ as $\Delta \to 0$. By using 1000 particles and 100 trajectories, we find out that the rate of mean-square stability gradually approaches to -4 as the stepsize decreases

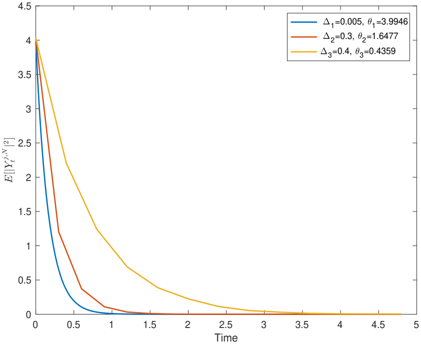

Figure: mean-square stability with three stepsizes

Long Time $\mathbb{W}_0$-$\widetilde{\mathbb{W}}_1$ type Propagation of Chaos

Let (E,ρ) be a Polish space and o be a fixed point in E. Let $\mathscr{P}(E)$ be the set of all probability measures on $\mathbb{R}^d$ equipped with the weak topology. For $p>0$, let

$$\mathscr{P}_p(E) := \{\mu \in \mathscr{P}(E) : \mu(\rho(o,\cdot)^p) < \infty\},$$

and define the L^p-Wasserstein distance

$$\mathbb{W}_p(\mu,\nu) = \inf_{\pi\in\mathbf{C}(\mu,\nu)} \left(\int_{E\times E} \rho(x,y)^p \pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p\vee 1}}, \quad \mu,\nu \in \mathscr{P}_p(E),$$

where $\mathbf{C}(\mu,\nu)$ is the set of all couplings of μ and ν. When $p>0$, $(\mathscr{P}_p(E),\mathbb{W}_p)$ is a polish space. To discuss the interacting particle system, we will consider the product space E^k with $k\geq 1$. For any $k\geq 1$, define

$$\rho_1(x,y) = \sum_{i=1}^k \rho(x^i,y^i), \quad x=(x^1,x^2,\cdots,x^k), y=(y^1,y^2,\cdots,y^k) \in E^k$$

and define

$$\widetilde{\mathbb{W}}_1(\mu,\nu) = \inf_{\pi\in\mathbf{C}(\mu,\nu)} \left(\int_{E^k\times E^k} \rho_1(x,y)\pi(\mathrm{d}x,\mathrm{d}y)\right), \quad \mu,\nu\in\mathscr{P}_1(E^k),$$

where $\mathscr{P}_1(E^k) = \{\mu\in\mathscr{P}(E^k) : \mu(\rho_1(\mathbf{o},\cdot)) < \infty\}$ for $\mathbf{o}=(o,o,\cdots,o)\in E^k$.

We will also use the total variation distance:

$$\|\gamma-\widetilde{\gamma}\|_{var}=\sup_{\|f\|_\infty\le 1}|\gamma(f)-\widetilde{\gamma}(f)|,\quad \gamma,\widetilde{\gamma}\in\mathscr{P}(E).$$

$$\|\gamma-\widetilde{\gamma}\|_{var}=\sup_{\|f\|_\infty\le 1, f\in C_b^2(\mathbb{R}^n)}|\gamma(f)-\widetilde{\gamma}(f)|,\quad \gamma,\widetilde{\gamma}\in\mathscr{P}(\mathbb{R}^n). \tag{27}$$

In addition, it holds

$$\|\gamma-\widetilde{\gamma}\|_{var}=2\mathbb{W}_0(\widetilde{\gamma},\gamma):=2\inf_{\pi\in\mathbf{C}(\gamma,\widetilde{\gamma})}\int_{\mathbb{R}^n\times\mathbb{R}^n}1_{\{x\neq y\}}\pi(\mathrm{d}x,\mathrm{d}y),\quad \gamma,\widetilde{\gamma}\in\mathscr{P}(\mathbb{R}^n).$$

Let Z_t be an n-dimensional Lévy process on some complete filtration probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Recall that for a general n-dimensional Lévy process, its characteristic function has the form

$$\mathbb{E}e^{i\langle \xi, Z_t\rangle} = \exp\left\{ i\langle \eta, \xi\rangle t - \frac{1}{2}\langle a\xi, \xi\rangle t + t\int_{\mathbb{R}^n - \{0\}} e^{i\langle z, \xi\rangle} - 1 - i\langle z, \xi\rangle 1_{\{|z|\leq 1\}}\nu(\mathrm{d}z) \right\}, \quad \xi \in \mathbb{R}^n,$$

where $\eta \in \mathbb{R}^n$, a is an $n \times n$ non-negative definite symmetric matrix, ν is the Lévy measure satisfying

$$\int_{\mathbb{R}^n} (1 \wedge |z|^2)\nu(\mathrm{d}z) < \infty.$$

Let $b : [0, \infty) \times \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^n$ are measurable and are bounded on bounded set. Let $N \geq 1$ be an integer and $(Z_t^i)_{1\leq i\leq N}$ be i.i.d. copies of Z_t. Consider the non-interacting particle system:

$$\mathrm{d}X_t^i = b_t(X_t^i, \mathscr{L}_{X_t^i})\mathrm{d}t + \sigma_t(X_{t-}^i)\mathrm{d}Z_t^i, \quad 1 \leq i \leq N, \tag{28}$$

and the mean field interacting particle system

$$\mathrm{d}X_t^{i,N} = b_t(X_t^{i,N}, \widehat{\mu}_t^N)\mathrm{d}t + \sigma_t(X_{t-}^{i,N})\mathrm{d}Z_t^i, \quad 1 \leq i \leq N, \tag{29}$$

where $\mathscr{L}_{X_t^i}$ is the distribution of X_t^i while $\widehat{\mu}_t^N$ stands for the empirical distribution of $(X_t^{i,N})_{1\leq i\leq N}$, i.e.

$$\widehat{\mu}_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}.$$

Note that (28) consists of N independent McKean-Vlasov SDEs, which are written as

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(X_{t-})\mathrm{d}Z_t. \tag{30}$$

When (30) and (29) are well-posed, for any $\mu \in \mathscr{P}(\mathbb{R}^d)$, let $P_t^* \mu$ be the distribution of the solution to (30) with initial distribution μ, and for any exchangeable $\mu^N \in \mathscr{P}((\mathbb{R}^d)^N)$, $1 \le k \le N$, $(P_t^{[k],N})^* \mu^N$ be the distribution of $(X_t^{i,N})_{1\le i\le k}$ from initial distribution μ^N. Moreover, let $\mu^{\otimes k}$ denote the k independent product of μ, i.e. $\mu^{\otimes k} = \prod_{i=1}^k \mu$. For any $1 \le k \le N$, let π_k be the projection from $(\mathbb{R}^d)^N$ to $(\mathbb{R}^d)^k$ defined by

$$\pi_k(x) = (x^1, x^2, \cdots, x^k), \quad x = (x^1, x^2, \cdots, x^N) \in (\mathbb{R}^d)^N.$$

Then it is not difficult to see that

$$(P_t^{[k],N})^* \mu^N = \{(P_t^{[N],N})^* \mu^N\} \circ (\pi_k)^{-1}, \quad 1 \le k \le N. \tag{31}$$

We assume that the initial distribution of (29) is exchangeable.

Let us recall some progress on the propagation of chaos. There are fruitful results in the case $Z_t^i = W_t^i$, n-dimensional Brownian motion.

- When $b_t(x,\mu) = \int_{\mathbb{R}^d} \widetilde{b}_t(x-y)\mu(\mathrm{d}y)$ for some function $\widetilde{b}$ Lipschitz continuous in spatial variable uniformly in time variable, the following paper adopts the synchronous coupling method to explore the quantitative propagation of chaos in strong convergence.

A.-S. Sznitman, Topics in propagation of chaos, *In "École d'Été de Probabilités de Sain-Flour XIX-1989", Lecture Notes in Mathematics* 1464, p. 165-251, Springer, Berlin, 1991.

H. McKean, A class of Markov processes associated with nonlinear parabolic equations, Proc. Nat. Acad. Sci., 56 (1967), 1907-1911.

When $n = d$, $\sigma = I_{d\times d}$, the entropy method is introduced in the following papers to derive the quantitative entropy-entropy propagation of chaos:

$$\mathrm{Ent}((P_t^{[k],N})^*\mu_0^N|(P_t^*\mu_0)^{\otimes k}) \le \frac{k}{N}\mathrm{Ent}(\mu_0^N|\mu_0^{\otimes N}) + \frac{ck}{N}, \quad t\in[0,T], 1\le k\le N, \tag{32}$$

here the relative entropy of two probability measures is defined as

$$\mathrm{Ent}(\nu|\mu) = \begin{cases} \nu(\log(\frac{\mathrm{d}\nu}{\mathrm{d}\mu})), & \nu \ll \mu; \\ \infty, & \text{otherwise.} \end{cases}$$

- D. Bresch, P.-E. Jabin, Z. Wang, Mean-field limit and quantitative estimates with singular attractive kernels, *Duke Math. J.* 172(2023), 2591-2641.
- P.-E. Jabin, Z. Wang, Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels, *Invent. Math.* 214(2018), 523-591.
- P.-E. Jabin, Z. Wang, Mean field limit and propagation of chaos for Vlasov systems with bounded forces, *J. Funct. Anal.* 271(2016), 3588-3627.

The idea of the entropy method is to derive the evolution of $\mathrm{Ent}((P_t^{[N],N})^*\mu_0^N|(P_t^*\mu_0)^{\otimes N})$ with respect to t from the Fokker-Planck-Kolmogorov equations for $(P_t^{[N],N})^*\mu_0^N$ and $(P_t^*\mu_0)^{\otimes N}$ respectively. The procedure relies on the chain rule of the Laplacian operator. Then (32) is obtained by the tensor property of relative entropy:

$$\mathrm{Ent}((P_t^{[k],N})^*\mu_0^N|(P_t^*\mu_0)^{\otimes k}) \le \frac{k}{N}\mathrm{Ent}((P_t^{[N],N})^*\mu_0^N|(P_t^*\mu_0)^{\otimes N}), \quad t\in[0,T], 1\le k\le N.$$

Recently, in the the paper

D. Lacker, L. Le Flem, Sharp uniform-in-time propagation of chaos, *Probab. Theory Related Fields* 187(2023), 443-480.

authors estimate $\mathrm{Ent}((P_t^{[k],N})^*\mu_0^N|(P_t^*\mu_0)^{\otimes k})$ directly and then derive the sharp rate $\frac{k^2}{N^2}$ instead of $\frac{k}{N}$ for entropy-entropy propagation of chaos in the case of Lipchitz or bounded interaction. Additionally, In the the paper

D. Lacker, Hierarchies, entropy, and quantitative propagation of chaos for mean field diffusions, *Probab. Math. Phys.* 4(2023), 377-432.

author shows the sharp long time entropy-entropy propagation of chaos, and the sharp long time propagation of chaos in $\mathbb{W}_2$-distance and total variation distance.

Authors in the page

A. Durmus, A. Eberle, A. Guillin, R. Zimmer, An elementary approach to uniform in time propagation of chaos, *Proc. Amer. Math. Soc.* 148(2020), 5387-5398.

develop the asymptotic reflection coupling to derive the long time $\widetilde{\mathbb{W}}_1$-$\widetilde{\mathbb{W}}_1$ type propagation of chaos:

$$\widetilde{\mathbb{W}}_1((P_t^{[k],N})^*\mu_0^N, (P_t^*\mu_0)^{\otimes k}) \leq c\varepsilon(t)\frac{k}{N}\widetilde{\mathbb{W}}_1(\mu_0^N, \mu_0^{\otimes N}) + c\frac{k}{\sqrt{N}}, \quad t \geq 0, 1 \leq k \leq N \tag{33}$$

for some constants $c > 0$ and $\lim_{t\to\infty}\varepsilon(t) = 0$.

We can also see the long time $\widetilde{\mathbb{W}}_1$-$\widetilde{\mathbb{W}}_1$ type propagation of chaos:

W. Liu, L. Wu, C. Zhang, Long-time behaviors of mean-field interacting particles related to Mckean-Vlasov equations, *Comm. Math. Phys.* 387(2021), 179-214.

The asymptotic reflection coupling is also applied to study the long time behavior of one-dimensional McKean-Vlasov SDEs with common noise in

J. Bao, J. Wang, Long time behavior of one-dimensional McKean-Vlasov SDEs with common noise, *arXiv:2401.07665*.

For the coupling, we may read the following papers:

M.F. Chen, S.F. Li, Coupling methods for multidimensional diffusion processes, Ann. Probab. 17(1989), 151-177.

A. Eberle, Reflection coupling and contraction rates for diffusions, Probab. Theory Relat. Fields 166(2016), 851-886.

F.Y. Wang, Asymptotic coupling by reflection and applications for non-linear monotone SPDES, Nonlinear Anal. 117(2015), 169-188.

Compared with the above significant progresses on propagation of chaos in Brownian motion noise case, there are fewer results on the propagation of chaos in general Lévy noise case. The following paper derives the long time $\widetilde{\mathbb{W}}_1$-$\widetilde{\mathbb{W}}_1$ type propagation of chaos (33) for interacting particle system driven by Lévy noise, where the asymptotic refined basic coupling is used.

M. Liang, M. B. Majka, J. Wang, Exponential ergodicity for SDEs and McKean-Vlasov processes with Lévy noise, *Ann. Inst. Henri Poincaré Probab. Stat.* 57(2021), 1665-1701.

However, to our knowledge, the quantitative propagation of chaos in relative entropy in the Lévy noise even in α-stable noise case is still open. The difficulty lies in that the chain rule for $-(-\Delta)^{\frac{\alpha}{2}}$ is not explicit so that the existing entropy method in seems unavailable in the α-stable noise case.

We are going to investigate the long time $\mathbb{W}_0$-$\widetilde{\mathbb{W}}_1$ type propagation of chaos for mean field interacting particle system driven by Lévy noise, which is not so difficult as the challenged entropy-$\mathbb{W}_2^2$ type propagations of chaos but also inspiring in this direction.

A general result on long time $\mathbb{W}_0$-$\widetilde{\mathbb{W}}_1$ type propagation of chaos

Let $b^{(0)}: \mathbb{R}^d \to \mathbb{R}^d$, $b^{(1)}: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^n$ be measurable and bounded on bounded set. Recall that $(Z_t^i)_{1 \le i \le N}$ are i.i.d. n-dimensional Lévy processes. Consider the mean field interacting particle system

$$\mathrm{d}X_t^{i,N} = b^{(0)}(X_t^{i,N})\mathrm{d}t + \frac{1}{N}\sum_{m=1}^{N} b^{(1)}(X_t^{i,N}, X_t^{m,N})\mathrm{d}t + \sigma(X_{t-}^{i,N})\mathrm{d}Z_t^i, \quad 1 \le i \le N, \tag{34}$$

and non-interacting particle system

$$\mathrm{d}X_t^{i} = b^{(0)}(X_t^{i})\mathrm{d}t + \int_{\mathbb{R}^d} b^{(1)}(X_t^{i}, y)\mathscr{L}_{X_t^i}(\mathrm{d}y)\mathrm{d}t + \sigma(X_{t-}^{i})\mathrm{d}Z_t^i, \quad 1 \le i \le N. \tag{35}$$

We assume that SDEs (34) and (35) are well-posed. Again, let $P_t^*\mu_0 = \mathscr{L}_{X_t^i}$ with $\mathscr{L}_{X_0^i} = \mu_0 \in \mathscr{P}(\mathbb{R}^d)$, which is independent of i. For simplicity, we denote $\mu_t = P_t^*\mu_0$. For any exchangeable $\mu^N \in \mathscr{P}((\mathbb{R}^d)^N)$, $1 \le k \le N$, $(P_t^{[k],N})^*\mu^N$ denotes the distribution of $(X_t^{i,N})_{1 \le i \le k}$ from initial distribution μ^N.

To derive the long time $\mathbb{W}_0$-$\widetilde{\mathbb{W}}_1$ type propagation of chaos, for any $s \geq 0$, consider the decoupled SDE

$$\mathrm{d}X_{s,t}^{i,\mu,z} = b^{(0)}(X_{s,t}^{i,\mu,z})\mathrm{d}t + \int_{\mathbb{R}^d} b^{(1)}(X_{s,t}^{i,\mu,z}, y)\mu_t(\mathrm{d}y)\mathrm{d}t + \sigma(X_{s,t-}^{i,\mu,z})\mathrm{d}Z_t^i, \quad t \geq s \tag{36}$$

with $X_{s,s}^{i,\mu,z} = z \in \mathbb{R}^d$. Let

$$P_{s,t}^{i,\mu} f(z) := \mathbb{E} f(X_{s,t}^{i,\mu,z}), \quad f \in \mathscr{B}_b(\mathbb{R}^d), z \in \mathbb{R}^d, i \geq 1, 0 \leq s \leq t.$$

We also assume that (36) is well-posed so that $P_{s,t}^{i,\mu}$ does not depend on i and we denote

$$P_{s,t}^{\mu} = P_{s,t}^{i,\mu}, \quad i \geq 1. \tag{37}$$

Moreover, for any $x = (x^1, x^2, \cdots, x^k) \in (\mathbb{R}^d)^k$, $F \in \mathscr{B}_b((\mathbb{R}^d)^k)$, and $(s_1, s_2, \cdots, s_k) \in [0,t]^k$ define

$$(P_{s_1,t}^{\mu} \otimes P_{s_2,t}^{\mu} \otimes \cdots \otimes P_{s_k,t}^{\mu})F(x) := \mathbb{E}F(X_{s_1,t}^{1,\mu,x^1}, X_{s_2,t}^{2,\mu,x^2}, \cdots, X_{s_k,t}^{k,\mu,x^k}). \tag{38}$$

In particular, we denote

$$(P_{s,t}^{\mu})^{\otimes k} F(x) := (P_{s,t}^{\mu} \otimes P_{s,t}^{\mu} \otimes \cdots \otimes P_{s,t}^{\mu})F(x), \quad 0 \leq s \leq t. \tag{39}$$

For simplicity, we write $P_t^{\mu} = P_{0,t}^{\mu}$. For any $F \in C^1((\mathbb{R}^d)^k)$, $1 \leq i \leq k$, $x = (x^1, x^2, \cdots, x^k) \in (\mathbb{R}^d)^k$, let $\nabla_i F(x)$ denote the gradient with respect to x^i.

Theorem: general result, X. Huang, F. Yang, Y.,

Let $\mu_0^N \in \mathscr{P}_1((\mathbb{R}^d)^N)$ be exchangeable and $\mu_0 \in \mathscr{P}_p(\mathbb{R}^d)$ for some $p \geq 1$. Assume that the following conditions hold.

(i) For any $1 \leq k \leq N$, $F \in C_b^2((\mathbb{R}^d)^k)$ with $\|F\|_\infty \leq 1$, $t \geq 0$, it holds

$$\begin{aligned}&\int_{(\mathbb{R}^d)^k} F(x)\{(P_t^{[k],N})^*\mu_0^N\}(\mathrm{d}x) - \int_{(\mathbb{R}^d)^k}\{(P_t^\mu)^{\otimes k}F\}(x)(\mu_0^N \circ \pi_k^{-1})(\mathrm{d}x)\\ &= \int_0^t \sum_{i=1}^k \int_{(\mathbb{R}^d)^N} \left\langle B_s^i(x), [\nabla_i (P_{s,t}^\mu)^{\otimes k} F](\pi_k(x))\right\rangle \{(P_s^{[N],N})^*\mu_0^N\}(\mathrm{d}x)\mathrm{d}s \qquad (40)\end{aligned}$$

with

$$B_s^i(x) = \frac{1}{N}\sum_{m=1}^N b^{(1)}(x^i, x^m) - \int_{\mathbb{R}^d} b^{(1)}(x^i, y)\mu_s(\mathrm{d}y), \quad x = (x^1, x^2, \cdots, x^N) \in (\mathbb{R}^d)^N.$$

(ii) There exists a measurable function $\varphi : (0,\infty) \to (0,\infty)$ with $\int_0^T \varphi(s)\mathrm{d}s < \infty, T > 0$ such that

$$|\nabla P_{r,t}^\mu f| \leq \varphi((t-r)\wedge 1)\|f\|_\infty, \quad f \in \mathscr{B}_b(\mathbb{R}^d), 0 \leq r < t. \qquad (41)$$

Theorem: Cont.

(iii) There exist an increasing function $g : (0,\infty) \to (0,\infty)$ and a decreasing $\Delta : (0,\infty) \to (0,\infty)$ with $\lim_{N\to\infty} \Delta(N) = 0$ such that

$$\begin{aligned}&\int_{(\mathbb{R}^d)^N} \left|B_s^1(x)\right| \{(P_s^{[N],N})^* \mu_0^N\}(\mathrm{d}x)\\ &\le g(s)\left\{\frac{1}{N}\widetilde{\mathbb{W}}_1(\mu_0^N, \mu_0^{\otimes N}) + \Delta(N)\{1 + \{\mu_0(|\cdot|^p)\}^{\frac{1}{p}}\}\right\}. \qquad (42)\end{aligned}$$

(iv) There exist functions $\varepsilon : (0,\infty) \to (0,\infty)$ with $\lim_{t\to\infty} \varepsilon(t) = 0$ and $\widetilde{\Delta} : (0,\infty) \to (0,\infty)$ with $\lim_{N\to\infty} \widetilde{\Delta}(N) = 0$ such that

$$\begin{aligned}&\widetilde{\mathbb{W}}_1((P_t^{[N],N})^* \mu_0^N, (P_t^* \mu_0)^{\otimes N})\\ &\le \varepsilon(t)\widetilde{\mathbb{W}}_1(\mu_0^N, \mu_0^{\otimes N}) + \{1 + \{\mu_0(|\cdot|^p)\}^{\frac{1}{p}}\}N\widetilde{\Delta}(N), \quad t \ge 0. \qquad (43)\end{aligned}$$

Moreover, there exists a constant $c_0 > 0$ such that

$$\sup_{t\ge 0}(P_t^* \mu_0)(|\cdot|^p) < c_0(1 + \mu_0(|\cdot|^p)). \qquad (44)$$

Then there exists a constant $c > 0$ independent of t and N such that

$$\begin{aligned}\|(P_t^{[k],N})^* \mu_0^N - (P_t^* \mu_0)^{\otimes k}\|_{var} &\le ck\varepsilon(t-1)\frac{\widetilde{\mathbb{W}}_1(\mu_0^N, \mu_0^{\otimes N})}{N}\\ &+ c\{1 + \{\mu_0(|\cdot|^p)\}^{\frac{1}{p}}\}k(\Delta(N) + \widetilde{\Delta}(N)), \quad 1 \le k \le N, t \ge 1.\end{aligned}$$

Application in Brownian motion noise case

Let

$$Z_t^i = (W_t^i, B_t^i), \quad i \geq 1,$$

where $\{W_t^i\}_{i\geq 1}$ are independent d-dimensional Brownian motions, $\{B_t^i\}_{i\geq 1}$ are independent n-dimensional Brownian motions and $\{W_t^i\}_{i\geq 1}$ is independent of $\{B_t^i\}_{i\geq 1}$. Let $\beta > 0$, $b^{(0)}$ and $b^{(1)}$ be defined in Section 2 and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^n$ be measurable and bounded on bounded set. Consider

$$\begin{aligned} \mathrm{d}X_t^{i,N} &= b^{(0)}(X_t^{i,N})\mathrm{d}t + \frac{1}{N}\sum_{m=1}^N b^{(1)}(X_t^{i,N}, X_t^{m,N})\mathrm{d}t \\ &\quad + \sqrt{\beta}\mathrm{d}W_t^i + \sigma(X_t^{i,N})\mathrm{d}B_t^i, \quad 1 \leq i \leq N, \end{aligned} \tag{45}$$

and independent McKean-Vlasov SDEs:

$$\mathrm{d}X_t^i = b^{(0)}(X_t^i)\mathrm{d}t + \int_{\mathbb{R}^d} b^{(1)}(X_t^i, y)\mathscr{L}_{X_t^i}(\mathrm{d}y)\mathrm{d}t + \sqrt{\beta}\mathrm{d}W_t^i + \sigma(X_t^i)\mathrm{d}B_t^i, \quad 1 \leq i \leq N. \tag{46}$$

(A) There exists a constant $K_\sigma > 0$ such that

$$\frac{1}{2}\|\sigma(x_1) - \sigma(x_2)\|_{HS}^2 \le K_\sigma |x_1 - x_2|^2, \quad x_1, x_2 \in \mathbb{R}^d. \tag{47}$$

$b^{(0)}$ is continuous and there exist $R > 0$, $K_1 \ge 0, K_2 > 0$ such that

$$\langle x_1 - x_2, b^{(0)}(x_1) - b^{(0)}(x_2)\rangle \le \gamma(|x_1 - x_2|)|x_1 - x_2| \tag{48}$$

with

$$\gamma(r) = \begin{cases} K_1 r, & r \le R; \\ \left\{-\frac{K_1+K_2}{R}(r-R) + K_1\right\} r, & R \le r \le 2R; \\ -K_2 r, & r > 2R. \end{cases}$$

Moreover, there exists $K_b \ge 0$ such that

$$|b^{(1)}(x,y) - b^{(1)}(\widetilde{x},\widetilde{y})| \le K_b(|x - \widetilde{x}| + |y - \widetilde{y}|), \quad x, \widetilde{x}, y, \widetilde{y} \in \mathbb{R}^d. \tag{49}$$

Theorem: Brownian motion case (X. Huang, F. Yang, Y.)

Assume **(A)**. Let $\mu_0 \in \mathscr{P}_{1+\delta}(\mathbb{R}^d)$ for some $\delta \in (0,1)$ and $\mu_0^N \in \mathscr{P}_1((\mathbb{R}^d)^N)$ be exchangeable. If

$$K_b < \frac{2\beta^2}{(K_2 - K_\sigma)\left(\int_0^\infty s e^{\frac{1}{2\beta}\int_0^s \{\gamma(v)+K_\sigma v\}\mathrm{d}v}\mathrm{d}s\right)^2}, \tag{50}$$

then there exists a positive constant c independent of k and N such that

$$\|(P_t^{[k],N})^*\mu_0^N - (P_t^*\mu_0)^{\otimes k}\|_{var}$$
$$\leq kce^{-ct}\frac{\widetilde{\mathbb{W}}_1(\mu_0^N, \mu_0^{\otimes N})}{N} + c\{1 + \{\mu_0(|\cdot|^{1+\delta})\}^{\frac{1}{1+\delta}}\}kN^{-\frac{\delta}{1+\delta}}, \quad 1 \leq k \leq N, t \geq 1.$$

Application in α-stable noise case

Recall that d-dimensional rotationally invariant α-stable process has Lévy measure

$$\nu^\alpha(\mathrm{d}z) = \frac{c_{d,\alpha}}{|z|^{d+\alpha}}\mathrm{d}z$$

for some constant $c_{d,\alpha} > 0$ and the generator $-(-\Delta)^{\frac{\alpha}{2}}$ is defined by

$$-(-\Delta)^{\frac{\alpha}{2}} f(x) = \int_{\mathbb{R}^d-\{0\}} \{f(x+z) - f(x) - \langle \nabla f(x), z\rangle 1_{\{|z|\le 1\}}\}\nu^\alpha(\mathrm{d}z), \quad f \in C_b^2(\mathbb{R}^d), \|f\|_\infty \le 1.$$

Let $n = d$ and $\sigma = I_{d\times d}$. The (34) and (35) reduce to

$$\mathrm{d}X_t^{i,N} = b^{(0)}(X_t^{i,N})\mathrm{d}t + \frac{1}{N}\sum_{m=1}^N b^{(1)}(X_t^{i,N}, X_t^{m,N})\mathrm{d}t + \mathrm{d}Z_t^i, \quad 1 \le i \le N,$$

and

$$\mathrm{d}X_t^i = b^{(0)}(X_t^i)\mathrm{d}t + \int_{\mathbb{R}^d} b^{(1)}(X_t^i, y)\mathscr{L}_{X_t^i}(\mathrm{d}y)\mathrm{d}t + \mathrm{d}Z_t^i, \quad 1 \le i \le N$$

respectively. We make the following assumptions.

(B1) The generator of Z_t^i is $-(-\Delta)^{\frac{\alpha}{2}}$ for some $\alpha \in (1,2)$.

(B2) $b^{(0)}$ is continuous. There exist $\ell_0 > 0$, $K_1 \ge 0, K_2 > 0$, $K_b \ge 0$ such that

$$\begin{aligned}&\langle x_1 - x_2, b^{(0)}(x_1) - b^{(0)}(x_2)\rangle \\ &\le K_1|x_1 - x_2|^2 1_{\{|x_1-x_2|\le \ell_0\}} - K_2|x_1-x_2|^2 1_{\{|x_1-x_2|>\ell_0\}},\end{aligned} \tag{51}$$

and

$$|b^{(1)}(x,y) - b^{(1)}(\widetilde{x}, \widetilde{y})| \le K_b(|x-\widetilde{x}| + |y - \widetilde{y}|), \quad x, \widetilde{x}, y, \widetilde{y} \in \mathbb{R}^d.$$

Theorem: α-stable noise case (X. Huang, F. Yang, Y.)

Assume **(B1)**-**(B2)**. Let $\mu_0 \in \mathscr{P}_{1+\delta}(\mathbb{R}^d)$ for some $\delta \in (0, \alpha - 1)$ and $\mu_0^N \in \mathscr{P}_1((\mathbb{R}^d)^N)$ be exchangeable. If

$$K_b < \frac{2c_1^2 K_2}{(1+c_1)^2},$$

then there exist positive constants c, λ such that

$$\begin{aligned} &\|(P_t^{[k],N})^* \mu_0^N - (P_t^* \mu_0)^{\otimes k}\|_{var} \\ &\le kce^{-\lambda t} \frac{\widetilde{\mathbb{W}}_1(\mu_0^N, \mu_0^{\otimes N})}{N} + c\{1 + \{\mu_0(|\cdot|^{1+\delta})\}^{\frac{1}{1+\delta}}\} k N^{-\frac{\delta}{1+\delta}}, \quad 1 \le k \le N, t \ge 1. \end{aligned} \tag{52}$$

Thanks A Lot !